

MCMC Algorithms

Markov chain definition

Markov chain: discrete time stochastic process

Definition: a series of random variables X_0, X_1, X_2, \dots (all valued over the same state space) with the “memoryless” **Markov property:**

$$p(X_i = x | X_0 = x_0, X_1 = x_1, \dots, X_{i-1} = x_{i-1}) = p(X_i = x | X_{i-1} = x_{i-1})$$

The set E of all possible values of X_i is called the **state space**

2 parameters:

- 1 initial distribution $p(X_0)$
- 2 transition probabilities $T(x, A) = p(X_i \in A | X_{i-1} = x)$

NB: only **homogeneous** Markov chains considered here:

$$p(X_{i+1} = x | X_i = y) = p(X_i = x | X_{i-1} = y)$$

Markov chains properties

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Property: a Markov chain is **aperiodic** if nothing induces periodic behavior of the trajectories

Stationary law & ergodic theorem

Definition: A probability distribution \tilde{p} is called **invariant law** (or **stationary law**) for a Markov chain if it verifies the following property:

$$\text{if } X_i \sim \tilde{p}, \text{ then } X_{i+j} \sim \tilde{p} \quad \forall j \geq 1$$

Remark: a Markov chain can admit several stationary laws

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Ergodic theorem (infinite space): A positive irreducible and recurrent Markov chain admits a single invariant probability distribution \tilde{p} and converges towards it

Markov chain example (discrete state space) – I

A Baby follows a Markov chain every minute with 3 states:

S sleep

E eat

D diaper change

⇒ its activity in 1min only depends on its current activity

Matrix of transition probabilities:

$$P = \begin{pmatrix} X_i/X_{i+1} & S & E & D \\ S & 0.9 & 0.05 & 0.05 \\ E & 0.7 & 0 & 0.3 \\ D & 0.8 & 0 & 0.2 \end{pmatrix}$$

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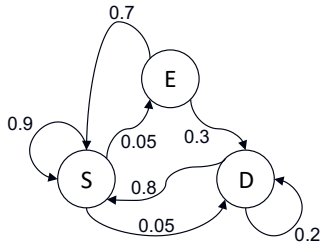
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- 1) Is the Markov chain irreducible ? recurrent ? aperiodic ?
- 2) Suppose Baby is now sleeping. What about in 2 min ? in 10 min ?
- 3) Suppose now that Baby is getting his/her diaper changed. What about in 10 min ?

Markov chain example (discrete state space) – II

1) Is the Markov chain irreducible ? recurrent ? aperiodic ?

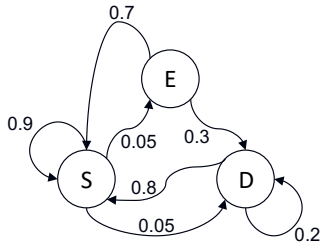


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Markov chain example (discrete state space) – II

1) Is the Markov chain irreducible ? recurrent ? aperiodic ?



2) Suppose Baby is now sleeping. What about in 2 min ? in 10 min ?

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}^T \quad \mathbb{P}(X_2|x_0) = x_0 P P = \begin{pmatrix} 0.885 \\ 0.045 \\ 0.070 \end{pmatrix}^T \quad \mathbb{P}(X_{10}|x_0) = x_0 P^{10} = \begin{pmatrix} 0.8839779 \\ 0.0441989 \\ 0.0718232 \end{pmatrix}^T$$

3) Suppose now that Baby is getting his/her diaper changed. What about in 10 min ?

Markov chain example (discrete state space) – II

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Markov chain example (discrete state space) – II

3) Suppose now that Baby his/her diaper changed. What about in 10 min ?

$$x'_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \quad \mathbb{P}(X_{10}|x'_0) = x'_0 P^{10} = \begin{pmatrix} 0.8839779 \\ 0.0441989 \\ 0.0718232 \end{pmatrix}^T$$

Here, the Markov chain being aperiodic, recurrent and irreducible, there is a stationary law: $\tilde{p} = \tilde{p}P$.